Homework 04

2. Winding number

> Suppose $f(z)$ is holomorphic in the disc $|z| \leq \epsilon$ and has a zero at $z = 0$ but nowhere else in the disc $|z| \leq \epsilon$. Show by direct integration that
>
> $$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{f'(z)}{f(z)} dz$$
>
> equals the winding number of the argument of f around the circle $|z| = \epsilon$. Then use the residue theorem to show that this equals the degree of the zero, in agreement with the argument principle.

The integral

$$\int_C \frac{f'(z)}{f(z)} dz = \int_C \frac{1}{f(z)} df(z) = \int_{f(C)} \frac{1}{w} dw$$

effectively measures the total change in the argument of $f(z)$ as z traverses the circle, which equals the winding number of the argument of f.

Since $f(z)$ is holomorphic in the disc $|z| \leq \epsilon$ and has a zero at $z = 0$, $f(z)$ can be locally expressed as $z^n g(z)$, where n is the degree of the zero.

Then, $f'(z) = nz^{n-1} g(z) + z^n g'(z)$, and so

$$\frac{f'(z)}{f(z)} = \frac{n}{z} + \frac{g'(z)}{g(z)}$$

The residue at $z = 0$ is the coefficient of $\frac{1}{z}$ in this expression, which is n, the degree of the zero.

4. Analytic continuation and Fourier coefficents

Give an analytic continuation of $\cos\theta$ from the unit circle $z = e^{i\theta}$ to the complex plane minus the origin.

Conclude that the Fourier coefficients c_n of $e^{-\cos\theta}$ decrease faster than any exponential, meaning $c_n = o(e^{-\alpha n})$ for all α as $n \to \pm\infty$. Compare this to the Fourier series of $1/(cos\theta - 3/2)$, what is the decay of its Fourier coefficents?

Analytic continuation

On the unit circle $z = e^{i\theta}$, $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + z^{-1})$. This expression provides an analytic continuation of $\cos\theta$ to the complex plane minus the origin, as it is well-defined for all $z \neq 0$.

Fourier coefficents of $e^{-\cos\ \theta}$

The function $e^{-\cos\ \theta}$ is smooth and periodic. The Fourier coefficients c_n of a periodic function $f(\theta)$ are given by:

$$c_n = 1/2\pi \int_0^{2\pi} f(\theta)e^{in\theta}d\theta = 1/2\pi \int_0^{2\pi} e^{-\cos\theta + in\theta}d\theta$$

$$1/2\pi \int_0^{2\pi} e^{-\cos\theta+in\theta}d\theta = 1/2\pi \int_{|z|=1} e^{-z/2-1/2z}z^n\frac{dz}{iz} = Res_{z=0}e^{-z/2-1/2z}z^{n-1}$$

The coefficient a_{-n} of $g(z) = e^{-z/2-1/2z}$ at the point of $z = 0$ can de derived by expanding by separately $e^{-z/2}$ as Talyor Series and $e^{-1/(2z)}$ as a Laurent Series and then multiplying these series together

$$a_{-n} = \sum_{k=0}^{\infty}(-\frac{1}{2})^k\frac{1}{k!}(-\frac{1}{2})^{(n+k)}\frac{1}{(n+k)!}$$

So fourier coefficients c_n

$$c_n = a_{-n} \leq |a_{-n}| \leq \frac{1}{2^n}\frac{1}{n!}$$

decrease faster than any exponential.

Fourier coefficents of $\frac{1}{\cos\ \theta - 3/2}$

The Fourier coefficients c_n of a periodic function $f(\theta)$ are given by:

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{in\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\cos\ \theta - 3/2}e^{in\theta}d\theta \\ &= \frac{1}{2\pi i}\int_{|z|=1} \frac{2z}{z^2-3z+1}z^{n-1}dz \end{aligned}$$

Let $z^2-3z+1=(z-z_1)(z-z_2)$, where $z_1=\frac{1}{2}\left(3-\sqrt{5}\right), z_2=\frac{1}{2}\left(3+\sqrt{5}\right)$.

$$\begin{aligned} c_n &= Res_{z=z_1}\frac{2z}{(z-z_1)(z-z_2)}z^{n-1} \\ &= \frac{2z_1}{z_1-z_2}{z_1}^{n-1} \end{aligned}$$

5. Laurent series and singularity

Let's consider a function f that is holomorphic in a disc around z_0 except at z_0 itself.

1. **Removable Singularity:**

   If f has a removable singularity at z_0, it means that f can be extended to a holomorphic function at z_0. In terms of the Laurent series, this implies that all the coefficients a_n for $n<0$ are zero because it reduces to its Taylor series.

   Conversely, if all $a_n=0$ for $n<0$, the Laurent series reduces to a Taylor series, implying that f is holomorphic at z_0 (since it can be expressed as a power series), and thus the singularity is removable.

2. **Pole of Order** m **:**

   If f has a pole of order m at z_0, it means that in the Laurent series, there is a term with $(z-z_0)^{-m}$ (where $a_{-m}\neq 0$) and no terms with higher negative powers.

   Conversely, if there is some $m<0$ such that $a_m\neq 0$ but $a_n=0$ for all $n<m$, then the Laurent series has a term $a_m(z-z_0)^m$ as its term with the highest negative power, indicating a pole of order m .

3. **Essential Singularity:**

   If the singularity at z_0 is neither removable nor a pole, it must be an essential singularity. This is characterized by the fact that there are infinitely many negative powers of $z - z_0$ in the Laurent series with non-zero coefficients. In other words, if the Laurent series has non-zero a_n for infinitely many $n < 0$, then z_0 is an essential singularity.

6. Euler proof of Basel problem

Using the result of the bonus problem, prove that

$$\sin \pi z = \prod (1 - z/n) e^{z/n} = \pi z \prod (1 - z^2/n^2)$$

Then compare the Taylor series of sin z to the first couple terms in the expansion of the infinite product to conclude

$$\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$$

According to the Weierstrass factorization theorem, an entire function can be represented as a product over its zeros. The function $\sin \pi z$ is entire and has zeros at all integers. The product representation for $\sin \pi z$ is given by:

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) \left(1 + \frac{z}{n}\right) = \sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

The Taylor series expansion of $\sin \pi z$ around $z = 0$ is:

$$\sin \pi z = \pi z - \frac{\pi^3 z^3}{3!} + \frac{\pi^5 z^5}{5!} - \frac{\pi^7 z^7}{7!} + \cdots$$

Now, let's expand the infinite product to the first couple of terms and keeping terms up to z^3, we get:

$$\begin{aligned} \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) &= \pi z \left(1 - \frac{z^2}{1^2}\right) \left(1 - \frac{z^2}{2^2}\right) \left(1 - \frac{z^2}{3^2}\right) \cdots \\ &= \pi z \left(1 - z^2 \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) + \cdots\right) \end{aligned}$$

Comparing the coefficient of z^3 from the Taylor series and the product expansion, we have:

$$-\frac{\pi^3}{6} = -\pi\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

This is the result we want.

Homework 4

1. Contour integrals

In[ ]:=
```
Integrate[z, {z, I, I + 1}]
```

Out[ ]=

$$\frac{1}{2} + i$$

In[ ]:=
```
Integrate[1 / (1 - z^2), {z, 0, I Infinity}]
```

Out[ ]=

$$\frac{i \pi}{2}$$

In[ ]:=
```
s1 = {z, -1 - I, -1 + I};
s2 = {z, -1 + I, 1 + I};
s3 = {z, 1 + I, 1 - I};
s4 = {z, 1 - I, -1 - I};
results = Map[Integrate[Abs[z] ^ 2, #] &, {s1, s2, s3, s4}]
Total@results
```

Out[ ]=

$$\left\{\frac{8 i}{3}, \frac{8}{3}, -\frac{8 i}{3}, -\frac{8}{3}\right\}$$

Out[ ]=

0

3. Laurent series

Use Mathematica to compute the Laurent series for

$f(z) = 1/(z(z - i)(z - 1))$

in the following annuli

$A_1 = \{z \mid 0 < |z| < 1\}$

$A_2 = \{z \mid 0 < |z - 1| < 1\}$

$A_3 = \{z \mid 0 < |z - i| < 1\}$

$A_4 = \{z \mid 1 < |z|\}$.

In[∘]:=
```
f[z_] := 1 / (z (z - I) (z - 1))
n = 5;
Series[f[z], {z, 0, n}]
Series[f[z], {z, 1, n}]
Series[f[z], {z, I, n}]
Series[f[z], {z, Infinity, n}]
```

Out[∘]=

$$-\frac{i}{z} - (1+i) - z - i\,z^3 - (1+i)\,z^4 - z^5 + O[z]^6$$

Out[∘]=

$$\frac{\frac{1}{2}+\frac{i}{2}}{z-1} - \left(\frac{1}{2}+i\right) + \left(\frac{1}{4}+\frac{5\,i}{4}\right)(z-1) - \frac{5}{4}\,i\,(z-1)^2 - \left(\frac{1}{8}-\frac{9\,i}{8}\right)(z-1)^3 + \left(\frac{1}{8}-i\right)(z-1)^4 - \left(\frac{1}{16}-\frac{15\,i}{16}\right)(z-1)^5 + O[z-1]^6$$

Out[∘]=

$$-\frac{\frac{1}{2}-\frac{i}{2}}{z-i} - \left(1+\frac{i}{2}\right) + \left(\frac{1}{4}-\frac{5\,i}{4}\right)(z-i) + \frac{5}{4}(z-i)^2 + \left(\frac{1}{8}+\frac{9\,i}{8}\right)(z-i)^3 - \left(1-\frac{i}{8}\right)(z-i)^4 - \left(\frac{1}{16}+\frac{15\,i}{16}\right)(z-i)^5 + O[z-i]^6$$

Out[∘]=

$$\left(\frac{1}{z}\right)^3 + \frac{1+i}{z^4} + \frac{i}{z^5} + O\left[\frac{1}{z}\right]^6$$