

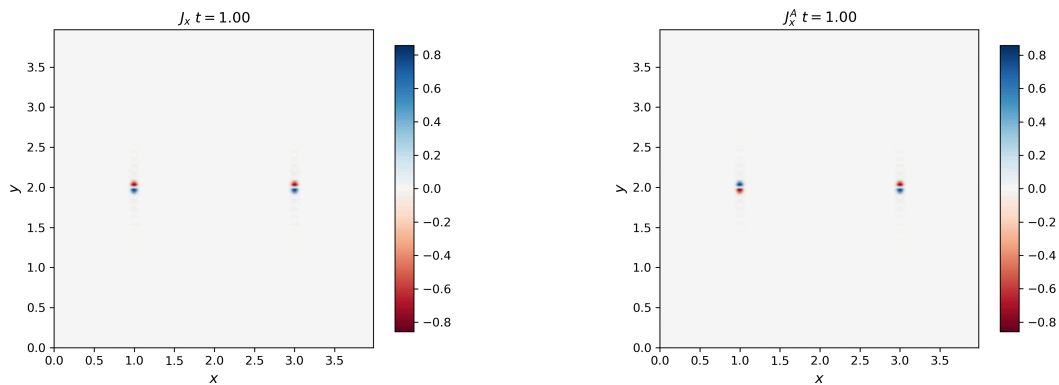
# Homework 05

## Alfvén current

Compute the parallel current from a *bipolar* Alfvén wavepacket. In the simulation lecture it was shown that a *unipolar* pulse (magnetic bomb, Bz unipolar gaussian) creates a field aligned current. The direction of that current was computed from curlB (left panel in the 2D result) and (in simulation coordinates) from  $J_x = [\pm 1/(\mu_0 C_a)] dE_y/dy$  (right panel in the 2D result).

## Unipolar Alfvén wavepacket

Multiply the RHS image by the sign corresponding to the propagation direction, we get the LHS image.



## Bipolar Alfvén wavepacket

To create a bipolar pulse in dBz and let it evolve on both sides. The init input is as follows:

```
case(6)
  !2D gaussian perturbation in Bz modulated by a odd function of x
  do iz = izmin,izmax
    do iy = iymin,iymax
      do ix = ixmin,ixmax
        uu(ix,iy,iz,7) = uu(ix,iy,iz,7) + db0 * (xgrid(ix) - 0.5*Lx) *
&
        exp( -( (xgrid(ix)-0.5*Lx)/(0.01*Lx) )**2 - ((ygrid(iy)-0.5*Ly)/
(0.01*Ly))**2)
```

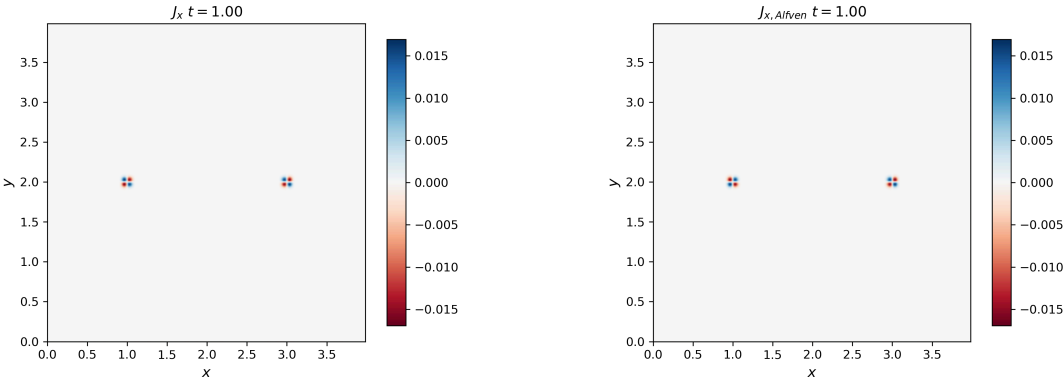
```

enddo
enddo
enddo

```

The basic idea is to change the amplitude modulation part from a constant to a odd function like  $(x_{grid}(ix) - 0.5*Lx)$  (sin would also work).

The figures below show the current density in the x direction for the bipolar Alfvén wavepacket.



The notebooks to reproduce the results are available here with initial input here.

### Hydrodynamic shock

Show that the entropy across a hydrodynamic shock increases: (i) First, prove equation III.80 in Siscoe (you can use the jumps in  $p$  and  $\rho$  previously derived). (ii) Next, take the derivative of the entropy ratio as function of Mach number and show it is positive. Thus explain that starting from  $M_1=1$  and moving upwards all  $M_1$  values have positive entropy ratios. (iii) Plot the entropy ratio as function of  $M_1$  for a range of adiabatic indices. (iv) Under what conditions does the entropy not increase? [Ans.  $\gamma = 1$ ] When might such conditions occur in space plasmas and why? [Ans. parallel shocks]

Given the jump conditions of pressure and density:

$$\rho_2 \rightarrow \frac{(\gamma + 1)}{\gamma + \frac{2}{M_1^2} - 1} \rho_1$$

$$p_2 \rightarrow \frac{(2\gamma M_1^2 - (\gamma - 1))}{\gamma + 1} p_1$$

And as the entropy is defined as  $\alpha_i := p_i / \rho_i^\gamma$

```
assums = { Subscript[ρ, 1] > 0, γ > 0, Subscript[M, 1] > 0};
ratio = Simplify[α[2]/α[1] /. rules, assums]
```

gives us

$$\alpha[2]/\alpha[1] = (\gamma + 1)^{-\gamma-1} \left( \gamma + \frac{2}{M_1^2} - 1 \right)^\gamma (-\gamma + 2\gamma M_1^2 + 1) = \left( \frac{1}{\gamma + 1} \right)^{\gamma+1} [2\gamma M_1^2 - (\gamma - 1)] \left( \gamma - 1 + \frac{2}{M_1^2} \right)^\gamma$$

Taking the derivative of the entropy ratio with respect to  $M_1$ :

```
dRatio = D[ratio, M1];
dRatio = Simplify[dRatio, assums];
dRatio /. formatRules
```

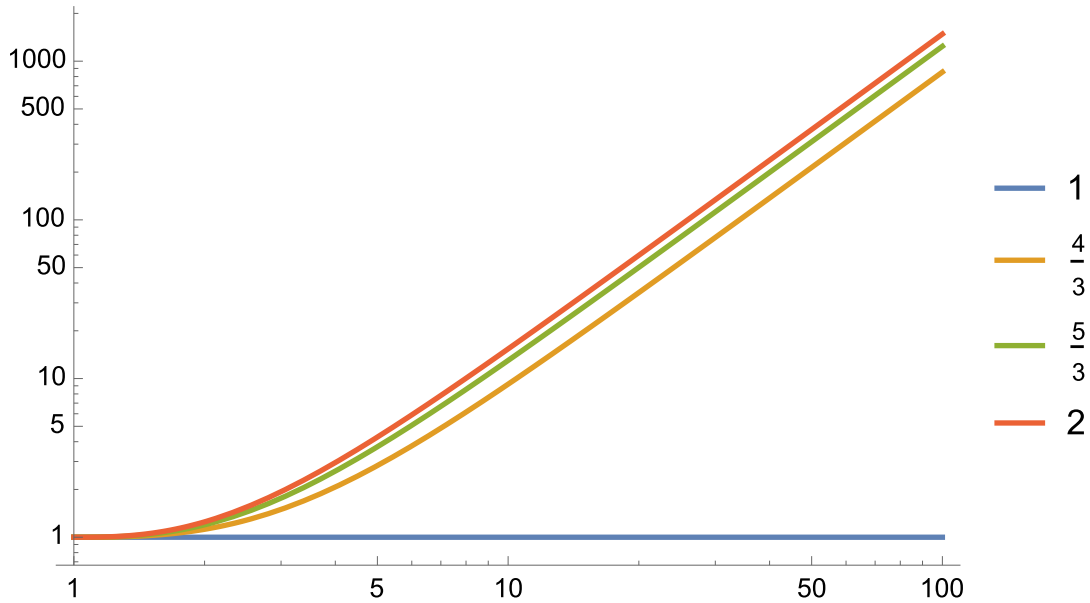
gives us

$$\frac{d}{dM_1} \left( \frac{\alpha_2}{\alpha_1} \right) = 4(\gamma - 1)\gamma(\gamma + 1)^{-\gamma-1} (M_1^2 - 1)^2 M_1^{-2\gamma-1} ((\gamma - 1)M_1^2 + 2)^{\gamma-1}$$

The derivative is positive for all  $M_1 > 0$  values as long as  $\gamma > 1$ . And when  $M_1 = 1$ , the ratio is equal to 1.

The plot of the entropy ratio as a function of  $M_1$  for a range of adiabatic indices is shown below.

```
γs = {1, 4/3, 5/3, 2};
LogLogPlot[
  Evaluate[ratio /. γ -> γs], {M1, 1, 100},
  PlotLegends -> γs
]
```



And we can see that when  $\gamma = 1$ , the entropy ratio is constant and equal to 1. As  $p = \alpha_1 \rho^\gamma = nkT$ , this corresponds to the case where the temperature is constant across the shock. Isothermal behavior is due to fast, field-aligned escaping electrons and this is more likely to occur in parallel shocks.

The mathematica notebook for this question and the following question is available [here](#).

## Perpendicular Shock

- a. Show Eq. III.90 following Siscoe notes. Follow same procedure as in III.73 to factor the known solution and derive the quadratic coefficients. Write  $(X-1)(X^2 + A_1 X + A_0) = 0$  where  $X = (U_2/U_1)$  and solve for  $A_0$  and  $A_1$  given the coefficients of the cubic. For extra testing of cubic, make sure that  $X=1$  is indeed a valid solution. As another test of your cubic equation (you can do that always during the derivation): Set  $A_1 \rightarrow \infty$  and validate it becomes a quadratic that is identical to III.71 (with  $M_1$  defined same as  $S_1$ ).

The continuity relations for the perpendicular shock can be written as

$$\begin{aligned}
 [[\rho U]] &= 0 \\
 \left[ \left[ \rho U^2 + p + \frac{B^2}{2\mu_0} \right] \right] &= 0 \\
 \left[ \left[ \left( \frac{1}{2} \rho U^2 + \frac{\gamma}{\gamma-1} \rho + \frac{B^2}{\mu_0} \right) U \right] \right] &= 0 \\
 [[UB]] &= 0
 \end{aligned}$$

Normalizing the magnetic pressure by the ram pressure and the thermal pressure by the ram pressure

$$B_1^2 \rightarrow \frac{\rho_1 U_1^2}{A^2}$$

$$p_1 \rightarrow \frac{\rho_1 U_1^2}{\gamma S^2}$$

And Eliminating  $B_2$ ,  $\rho_2$  and  $p_2$  from the equations

```
massEq := \[Rho]1 U1 == \[Rho]2 U2
momEq := \[Rho]1 U1^2 + p1 + B1^2/2 == \[Rho]2 U2^2 + p2 + B2^2/2
energyEq := (1/2 \[Rho]1 U1^2 + f p1 + B1^2) U1 == (1/2 \[Rho]2 U2^2 +
f p2 + B2^2) U2
FaradayEq := B1 U1 == B2 U2
eqs = {massEq, momEq, energyEq, FaradayEq};

Simplify[
Eliminate[eqs, {B2, \[Rho]2, p2}] /. rules, {Subscript[U, 1] > 0,
Subscript[\[Rho], 1] > 0}]
```

We get the following equation

$$\frac{(r-1)(A^2 r (S^2(-\gamma + \gamma r + r + 1) - 2) + S^2(\gamma + \gamma(-r) - 2))}{A(\gamma - 1)S} = 0$$

where  $r = U_2/U_1$ .

Clearly,  $r = 1$  is a solution. We can collect the rest equation to get the quadratic equation in  $r$

```
Collect[S^2 (-2 + \[Gamma] - r \[Gamma]) + A^2 r (-2 + S^2 (1 + r - \[Gamma] +
r \[Gamma])) == 0, r]
```

we get

$$r^2(A^2\gamma S^2 + A^2 S^2) + r(-A^2\gamma S^2 + A^2 S^2 - 2A^2 - \gamma S^2) + \gamma S^2 - 2S^2 = 0$$

This is exactly the same as Eq. III.90 in Siscoe notes.

- b. Solve the quadratic to find the two solutions for  $U_2$ . Show that one is non-physical, and identify the remaining, physical one.

```
assums = {A > 0 , S > 0, \[Gamma] > 1};
Solve[eqR, {r}, Reals, Assumptions -> assums] // Simplify
```

Solve the quadratic equation, we get

$$\left. \frac{-(\gamma - 1)S^2 + 2)^2 + A^2((-2\gamma^2 + 2\gamma + 8)S^4 + 4\gamma S^2) + \gamma^2 S^4 + \gamma S^2}{2A^2(\gamma + 1)S^2} \right\}, \left\{ r \rightarrow \frac{A^2((\gamma - 1)S^2 + 2) + \sqrt{A^4((\gamma - 1)S^2 + 2)^2 + A^2((-2\gamma^2 + 2\gamma + 8)S^4 + 4\gamma S^2) + \gamma^2 S^4 + \gamma S^2}}{2A^2(\gamma + 1)S^2} \right.$$

The first solution is non-physical as it is negative. The physical solution is

$$r_1 := \frac{A^2((\gamma - 1)S^2 + 2) + \sqrt{A^4((\gamma - 1)S^2 + 2)^2 + A^2((-2\gamma^2 + 2\gamma + 8)S^4 + 4\gamma S^2) + \gamma^2 S^4 + \gamma S^2}}{2A^2(\gamma + 1)S^2}$$

In the limit of  $A \rightarrow \infty$

```
Subscript[r, 1] :=
  1/(2 A^2 S^2 (1 + \[Gamma])) (A^2 (2 + S^2 (-1 + \[Gamma])) +
    S^2 \[Gamma] + \[Sqrt](A^4 (2 + S^2 (-1 + \[Gamma]))^2 +
      S^4 \[Gamma]^2 +
      A^2 (4 S^2 \[Gamma] + S^4 (8 + 2 \[Gamma] - 2 \[Gamma]^2))))
Simplify[Limit[Subscript[r, 1], {A -> \[Infinity]}], assums]
```

we can see that the solution becomes

$$\frac{(\gamma - 1)S^2 + 2}{(\gamma + 1)S^2}$$

which is the same as Eq. III.71 in Siscoe notes for hydrodynamic shocks.

- c. Plot the solution as function of  $A_1$  and  $M_1$  (color, contour or surface plot OK). Show that the limits for  $A_1 \rightarrow \infty$  and  $M_1 \rightarrow \infty$  are as expected.

In the limit of  $A_1 \rightarrow \infty$  and  $M_1 \rightarrow \infty$ , the solution approaches  $\frac{\gamma-1}{\gamma+1}$ . For  $\gamma = 5/3$ , the solution approaches 0.25, which is expected. The contour plot of the solution is shown below.

```
Block[{\[Gamma] = 5/3, p1, p2},
  p1 = ContourPlot[r1, {A, 0, 100}, {S, 0, 100},
    PlotLegends -> Automatic];
  p2 = ContourPlot[r1, {A, 0, 10}, {S, 0, 10},
    PlotLegends -> Automatic];
  Export["figures/shock_U2.svg", p1];
  Export["figures/shock_U2_zoom.svg", p2];
  {p1, p2}
]
```

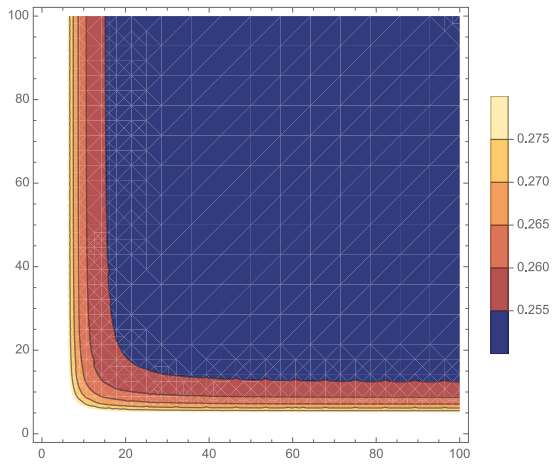


Figure 1: Contour plot with A and S less than 100

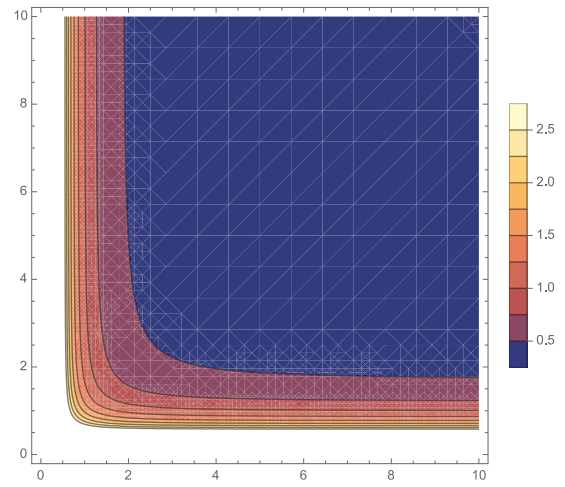


Figure 2: Zoom in plot for A and S less than 10

## Bibliography